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THE INTEGRATION OF POISSON'S EQUATIONS IN THE CASE OF THREE LINEAR INVARIANT RELATIONS†

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The conditions for Poisson's equations to be integrable in the case of three linear invariant relations of the equations of motion of a body in a field of potential and gyroscopic forces [1–3] are investigated. New ways of integrating Poisson's equations are obtained, which correspond to the case when a fractionally linear first integral exists in these equations. © 2002 Elsevier Science Ltd. All rights reserved.

The construction of new solutions of the equations of the dynamics of a rigid body [4, 5] enables information to be obtained on the properties of integral manifolds of non-linear differential equations and enables investigations to be made of the motion of a body that are necessary for mechanics. In the problem of the motion of a body under the action of potential and gyroscopic forces, which is described by Kirchhoff-class equations [1, 3], solutions with three invariant relations that are linear with respect to the main variable of the problem have been investigated in most detail [1, 2]. However, the version when the integrals of the energy and angular momentum of the motion in these relations becomes a consequence of the geometric integral, has not been completely considered. This is due to the fact that Poisson's equations have only one first integral and hence, in general, their integration does not reduce to quadratures [2]. Special cases of the integration of these equations have been considered previously in [2, 6].‡

1. FORMULATION OF THE PROBLEM

We will write the differential equations of the motion of a gyrostat with a fixed point acted upon by potential and gyroscopic forces [1–3] in the previously used notation [6]

$$\dot{\mathbf{x}} = (\mathbf{x}+\boldsymbol{\lambda})\times a\mathbf{x} + a\mathbf{x}\times B\boldsymbol{\nu} + \mathbf{s}\times\boldsymbol{\nu} + \boldsymbol{\nu}\times C\boldsymbol{\nu}, \quad \dot{\boldsymbol{\nu}} = \boldsymbol{\nu}\times a\mathbf{x} \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ is the angular momentum of the gyrostat, $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ is the unit vector of the axis of symmetry of the force field, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the gyrostatic moment, characterizing the motion of the carried bodies, $\mathbf{s} = (s_1, s_2, s_3)$ is a vector codirectional with the vector of the generalized centre of mass, $a = (a_{ij})$ $(i, j = 1, 2, 3)$ is the gyration tensor, constructed at a fixed point, $B = (B_{ij})$ is a third-order constant symmetrical matrix, which defines the gyroscopic forces, and $C = (C_{ij})$ is a third-order constant symmetrical matrix, characterizing the potential forces. The dot above the variables denotes a derivative with respect to time t.

Equations (1.1) have first integrals

$$\mathbf{x}\cdot a\mathbf{x} - 2(\mathbf{s}\cdot\boldsymbol{\nu}) + C\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 2E, \quad \boldsymbol{\nu}\cdot(\mathbf{x}+\boldsymbol{\lambda}) - \tfrac{1}{2}(B\boldsymbol{\nu}\cdot\boldsymbol{\nu}) = k, \quad \boldsymbol{\nu}\cdot\boldsymbol{\nu} = 1 \tag{1.2}$$

Here E and k are arbitrary constants.

Equations (1.1) in other variables [3] describe the motion of a body in an ideal incompressible fluid [1, 2] and can be integrated as Kirchhoff-class differential equations. In view of this, the construction of new solutions of Eqs (1.1) also leads to the construction of new solutions in the problem of the motion of a body in a fluid.

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‡See also: MOZALEVSKAYA, G. V. and LESINA, M. Ye., Linear invariant relations of Kirchhoff's equations. Preprint No 01.02, Nats. Akad. Nauk Ukrainy Inst. Prikl. Matem. i Mekh., Donetsk, 2001.

We will consider the integration of Eqs (1.1) under the condition that they allow of three linear invariant relations (everywhere henceforth summation over the corresponding subscript is carried out from 1 to 3)

$$x_1 = b_0 + \sum b_i \nu_i, \quad x_2 = c_0 + \sum c_j \nu_j, \quad x_3 = d_0 + \sum d_k \nu_k \tag{1.3}$$

where b_n, c_n and d_n ($n = 0, 1, 2, 3$) are constants.

The conditions for these relations to exist in different cases have been investigated in [1, 2, 6] and in the reference cited in the previous footnote.

When integrating the equations corresponding to the second equality of (1.1) when relations (1.3) hold, i.e. the equations

$$\dot{\nu}_1 = a_3 \nu_2 (d_0 + \sum d_k \nu_k) - a_2 \nu_3 (c_0 + \sum c_j \nu_j) \quad (1\ 2\ 3,\ bcd) \tag{1.4}$$

(everywhere henceforth it is assumed that the two unwritten relations are obtained from cyclic permutation of the symbols given in parenthesis), difficulties arise related to the version when, substitution of relations (1.3) into the first two integrals of (1.2) leads to the equality $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$. In this case Eqs (1.4) have only a single first integral.

Equations (1.4) were written in the principal system of coordinates in which $a_{ij} = 0$ ($i \neq j$), $a_{ii} = a_i$ ($i = 1, 2, 3$).

Chaplygin [2], when considering the problem of the motion of a body in a fluid when relations (1.3) occur in the equations of motion, was the first to point out that the integration of Eqs (1.4) in quadratures is extremely difficult. Kharlamov [1], when investigating relations (1.3), excluded the case when the first integrals are degenerate.

The conditions connecting the parameters of problem (1.1), in satisfying which the substitution of relations (1.3) into the scalar equations which follow from the first equality of (1.1), taking the second equality of (1.1) into account, leads to identities in the variables ν_1, ν_2 and ν_3, while the substitution of expressions (1.3) into the first two integrals of (1.2) gives a geometric integral, have the form

$$\begin{gathered} b_0 = -\lambda_1, \quad b_1 = -\frac{1}{2}(B_{22} + B_{33}), \quad c_1 + b_2 = B_{12} \quad (1\ 2\ 3,\ bcd) \\ s_i = -(a_1 \lambda_1 b_i + a_2 \lambda_2 c_i + a_3 \lambda_3 d_i), \quad i = 1, 2, 3 \\ C_{ij} = -(a_1 b_i b_j + a_2 c_i c_j + a_3 d_i d_j), \quad i \neq j \\ C_{ii} = -(a_1 b_i^2 + a_2 c_i^2 + a_3 d_i^2), \quad i, j = 1, 2, 3 \end{gathered} \tag{1.5}$$

Hence, it is only necessary to integrate Eqs (1.4). After integrating these equations, we find from relations (1.3) the time-dependences of the components x_i, while the components of the angular velocity $\boldsymbol{\omega} = a\mathbf{x}$ are found from the formulae

$$\omega_i = a_i x_i, \quad i = 1, 2, 3 \tag{1.6}$$

2. INTEGRATION OF SYSTEM (1.4) USING THE GENERALIZED ZHUKOVSKII INTEGRAL [7]

Suppose invariant relations (1.3) have the form

$$x_1 = -\lambda_1 + b_1 \nu_1 \quad (1\ 2\ 3,\ bcd)$$

i.e. some of the conditions, imposed on the parameters, are simplified

$$s_i = -a_i \lambda_i b_i; \quad B_{ij} = 0, \quad i \neq j; \quad C_{ij} = 0, \quad i \neq j; \quad i, j = 1,2,3 \tag{2.1}$$

$$C_{11} = -a_1 b_1^2 \quad (1\ 2\ 3,\ bcd)$$

Then system (1.4) takes the symmetric form

$$\dot{\nu}_1 = a_2 \lambda_2 \nu_3 - a_3 \lambda_3 \nu_2 + (a_3 d_3 - a_2 c_2) \nu_2 \nu_3 \quad (1\ 2\ 3,\ bcd) \tag{2.2}$$

and has two integrals

$$\sum \nu_i^2 = 1, \quad a_1 b_1 \nu_1^2 + a_2 c_2 \nu_2^2 + a_3 d_3 \nu_3^2 - 2\sum a_i \lambda_i \nu_i = C_0 \tag{2.3}$$

where C_0 is an arbitrary constant.

System (2.2) can be formally compared with the system

$$\dot{\omega}_1 = \frac{A_2 - A_3}{A_1}\omega_2\omega_3 + \frac{1}{A_1}(\lambda_2\omega_3 - \lambda_3\omega_2) \quad (1\ 2\ 3) \tag{2.4}$$

which describes the Zhukovskii case [7] of the problem of the motion of a heavy gyrostat. Here A_i are the principal moments of inertia and ω_i are the components of the angular velocity vector, $i = 1, 2, 3$. Since $A_i > 0$, the Zhukovskii integral is such that

$$\sum (A_i\omega_i + \lambda_i)^2 = x_0^2 \tag{2.5}$$

where x_0 is an arbitrary constant.

In the case (2.1)–(2.3) we can conclude from the physical meaning of the quantities B_{11}, B_{22} and B_{33} that the parameters b_1, c_2 and d_3 can be have both positive and negative values.

Integral (2.2), in general, can be converted into a function which describes a second-order central surface (an ellipsoid or unparted and parted hyperboloids). Consequently, only when the second relation of (2.3) corresponds to the equation of an ellipsoid is the procedure for reducing the integration of system (2.2) to quadratures the same as for system (2.4) with integral (2.5).

Consider the following example: $\lambda_3 = 0$, $a_* = a_3d_3 - a_1b_1 > 0$, $b_* = a_2c_2 - a_3d_3 > 0$. The second integral of (2.3), using the first integral of (2.3), can be represented in the form

$$a_*(\nu_1 + a_*^{-1}a_1\lambda_1)^2 - b_*(\nu_2 - b_*^{-1}a_2\lambda_2)^2 = E_0 \tag{2.6}$$

where E_0 is an arbitrary constant.

Using parametrization of relation (2.6)

$$\begin{aligned} \nu_1 &= \nu_1(u) = a_*^{-1/2}\left(\sqrt{E_0}\,\mathrm{ch}\,u - a_*^{-1/2}a_1\lambda_1\right) \\ \nu_2 &= \nu_2(u) = b_*^{-1/2}\left(\sqrt{E_0}\,\mathrm{sh}\,u + b_*^{-1/2}a_2\lambda_2\right) \end{aligned} \tag{2.7}$$

we have from the geometric integral $\sum \nu_i^2 = 1$ and system (2.2)

$$\nu_3^2 = 1 - \nu_1^2(u) - \nu_2^2(u), \quad \dot{u} = -(a_*b_*)^{1/2}\left(1 - \nu_1^2(u) - \nu_2^2(u)\right)^{1/2} \tag{2.8}$$

Relations (2.7) and (2.8) define an integral manifold of system (2.2).

3. REDUCTION OF SYSTEM (1.4) TO VECTOR FORM. THE SYMMETRIC CASE

We will investigate system (1.4), basing ourselves on its vector form

$$\dot{\boldsymbol{\nu}} = \mathbf{m} \times \boldsymbol{\nu} + \boldsymbol{\nu} \times G^+\boldsymbol{\nu} + \boldsymbol{\nu} \times G^-\boldsymbol{\nu} \tag{3.1}$$

where

$$\mathbf{m} = (m_1, m_2, m_3) = (a_1\lambda_1, a_2\lambda_2, a_3\lambda_3), \quad G^\pm = (g_{ij}^\pm) \tag{3.2}$$

$$\begin{gathered} g_{11}^+ = 0, \quad g_{22}^+ = a_2c_2 - a_1b_1, \quad g_{33}^+ = a_3d_3 - a_1b_1, \quad g_{11}^- = g_{22}^- = g_{33}^- = 0 \\ g_{12}^\pm = \pm g_{21}^\pm = g_1^\pm, \quad g_{13}^\pm = \pm g_{31}^\pm = g_2^\pm, \quad g_{23}^\pm = \pm g_{32}^\pm = g_3^\pm \\ g_1^\pm = \frac{a_1b_2 \pm a_2c_1}{2}, \quad g_2^\pm = \frac{a_1b_3 \pm a_3d_1}{2}, \quad g_3^\pm = \frac{a_2c_3 \pm a_3d_2}{2} \end{gathered} \tag{3.3}$$

The structure of the third term in Eq. (3.1) enables us to reduce (3.1) to the form

$$\dot{\boldsymbol{\nu}} = \mathbf{m} \times \boldsymbol{\nu} + \boldsymbol{\nu} \times G^+ \boldsymbol{\nu} + \mathbf{n}(\boldsymbol{\nu} \cdot \boldsymbol{\nu}) - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}) \tag{3.4}$$

$$\mathbf{n} = (n_1, n_2, n_3); \quad n_1 = -g_3^-, \quad n_2 = g_2^-, \quad n_3 = -g_1^-$$

We will consider the case when $G^- = 0$. By virtue of relations (3.3) in system (1.5), the conditions imposed on b_0, b_1 (1 2 3, bcd), s_i and C_{ij} (i, j = 1, 2, 3) do not change, while the remaining ones give the equations

$$b_2 = \frac{a_2 B_{12}}{a_1 + a_2}, \quad b_3 = \frac{a_3 B_{13}}{a_1 + a_3} \quad (1\ 2\ 3, bcd)$$

When these constraints on the parameters of the problem are satisfied, Eq. (3.1) allows of two integrals

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1, \quad G^+ \boldsymbol{\nu} \cdot \boldsymbol{\nu} - 2(\mathbf{m} \cdot \boldsymbol{\nu}) = \varepsilon_0$$

where ε_0 is an arbitrary constant. Consequently, the problem of integrating Eq. (3.1) is reduced to quadratures. The angular velocity vector of the gyrostat, by virtue of Eqs (1.6) and (3.3), has the form

$$\boldsymbol{\omega} = -\mathbf{m} + G^+ \boldsymbol{\nu} \tag{3.5}$$

i.e. it contains only the symmetric matrix G^+.

4. THE CASE $G^+ = 0$, $\mathbf{m} = 0$

In Eqs (3.2) and (3.3) we will put $G^+ = 0$ and $\mathbf{m} = 0$. We then have from relations (1.5), (3.2) and (3.3)

$$b_2 = \frac{a_2 B_{12}}{a_2 - a_1}, \quad b_3 = \frac{a_3 B_{13}}{a_3 - a_1} \ (123, bcd), \quad c_2 = \frac{a_1 b_1}{a_2}, \quad d_3 = \frac{a_1 b_1}{a_3} \tag{4.1}$$

$$s_i = 0, \ i = 1,2,3; \ B_{11} = \varkappa_0 (a_2 a_3 - a_1 a_3 - a_1 a_2) \ (123), \quad \varkappa_0 = b_1 a_2^{-1} a_3^{-1}$$

In this case the form of the quantities C_{ij} (i, j = 1, 2, 3) from relations (1.5) does not change. The angular velocity vector of the gyrostat, unlike (3.5), is

$$\boldsymbol{\omega} = \mathbf{n} \times \boldsymbol{\nu} \tag{4.2}$$

It was shown in [6] that the conditions for Eqs (1.4) to be integrable are closely related to the conditions for isoconic motions of the body to exist in the case of three invariant relations (1.3). Isoconic motions of a body possess the following properties: the mobile and fixed hodographs of the velocity are symmetric to one another about the plane tangential to them. These motions can be characterized analytically by the invariant relations [4]

$$\boldsymbol{\omega} \cdot (\boldsymbol{\nu} - \mathbf{e}) = 0 \tag{4.3}$$

where $\boldsymbol{\omega}$ is the angular velocity and $\mathbf{e}$ is the unit vector, permanently connected with the body. Isoconic motions are of considerable importance in the kinematic interpretation of the motion of a body by the hodograph method [8]. For the case of the three linear invariant relations (1.3), the vector $\mathbf{e}$ must satisfy the vector equation [6]

$$a_1 b_1 \mathbf{e} + \mathbf{e} \times \mathbf{n} = -\mathbf{m} \tag{4.4}$$

If a solution of this equation exists for the vector $\mathbf{e}$ and $|\mathbf{e}| = 1$, the condition for motion (4.3) to be isoconic will be satisfied.

Consider Eq. (3.4) with $G^+ = 0$ and $\mathbf{m} = 0$

$$\dot{\boldsymbol{\nu}} = \mathbf{n}(\boldsymbol{\nu} \cdot \boldsymbol{\nu}) - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}) \tag{4.5}$$

For convenience we will change to dimensionless time $\tau = |\mathbf{n}| t$. Then, denoting differentiation with respect to τ by a prime, we have from Eq. (4.5)

$$\boldsymbol{\nu}' = \mathbf{n}_0 - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}_0); \ \mathbf{n}_0 = \mathbf{n}/|\mathbf{n}| = (n_0^{(1)}, n_0^{(2)}, n_0^{(3)}) \tag{4.6}$$

where we have taken into account the relation $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$.

We will make the following replacement of variables in Eq. (4.6)

$$\nu_l = n_0^{(l)} x - \frac{n_0^{(3-l)}}{n_*} y - \frac{n_0^{(l)} n_0^{(3)}}{n_*} z, \ \ l = 1,2; \ \ \nu_3 = n_0^{(3)} x + n_* z \tag{4.7}$$

$$n_* = \left[\left(n_0^{(1)}\right)^2 + \left(n_0^{(2)}\right)^2\right]^{1/2}$$

We substitute expressions (4.7) into the scalar equations which follow from Eq. (4.6)

$$x' = 1 - x^2, \ \ y' = -xy, \ \ z' = -xz \tag{4.8}$$

System (4.8) is easily integrated

$$x = \operatorname{th}(\tau + \tau_0), \ \ y = \frac{z}{c_*} = \frac{1}{\sqrt{1 + c_*^2} \operatorname{ch}(\tau + \tau_0)}; \ \ \operatorname{th} \tau_0 = x_0 \tag{4.9}$$

where τ_0 and c_* are arbitrary constants. Substituting expression (4.9) into (4.7), we obtain the relations $\nu_i = \nu_i(\tau)$, which, using Eqs (1.3), enable us to obtain $x_i = x_i(\tau)$, i.e. to solve the problem of integrating Eqs (1.1) completely.

The following property of Eqs (4.5) is of interest. On the basis of system (4.8) and the replacement of variables (4.7) it can be established that Eq. (4.5) has, in addition to the integral $\Sigma \nu_i^2 = 1$, an additional fractionally linear first integral

$$\frac{n_1 n_3 \nu_1 + n_2 n_3 \nu_2 - (n_1^2 + n_2^2)\nu_3}{n_1 \nu_2 - n_2 \nu_1} = L_0 \tag{4.10}$$

where L_0 is an arbitrary constant. This fact also enabled us to integrate Eq. (4.6) completely.

We will investigate the conditions for the motion of the gyrostat to be isoconic in this case. From Eq. (4.4) we conclude that $b_1 = 0$ and $\mathbf{e} = \mathbf{n}_0$. Consequently, in Eqs (4.1) we must put $B_{ii} = 0$ ($i = 1, 2, 3$) while in (1.3) we must put $c_2 = d_3 = 0$. Invariant relations (1.3) then take the form

$$x_1 = b_2 \nu_2 + b_3 \nu_3, \ \ x_2 = -\frac{a_1 b_2}{a_2} \nu_1 + c_3 \nu_3, \ \ x_3 = -\frac{a_1 b_3}{a_3} \nu_1 - \frac{a_2 c_3}{a_3} \nu_2$$

5. THE CONDITIONS FOR A FRACTIONALLY LINEAR FIRST INTEGRAL OF SYSTEM (3.4) TO EXIST WHEN $G^+ = 0$, $\mathbf{m} \neq 0$

Since, when $G^+ = 0$ and $\mathbf{m} = 0$ we have established that Poisson's equations allow of the fractionally linear integral (4.10), it is of interest to investigate the form of the integral in these equations when $G^+ = 0$, $\mathbf{m} \neq 0$.

For the equations

$$\dot{\boldsymbol{\nu}} = \mathbf{m} \times \boldsymbol{\nu} + \mathbf{n}(\boldsymbol{\nu} \cdot \boldsymbol{\nu}) - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}) \tag{5.1}$$

we specify the integral

$$\frac{\alpha_0 + (\boldsymbol{\alpha} \cdot \boldsymbol{\nu})}{\beta_0 + (\boldsymbol{\beta} \cdot \boldsymbol{\nu})} = l_0 \tag{5.2}$$

where l_0 is an arbitrary constant, α_0 and β_0 are fixed constants, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant vectors which satisfy the condition $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$. The last condition is best used at the stage when the problem is formulated, since the presence in the expansion of the vector $\boldsymbol{\alpha}$ of a component parallel to the vector $\boldsymbol{\beta}$ enables one, by a trivial transformation of integral (5.2), to reduce it to a form in which $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$.

We will introduce the following notation

$$\boldsymbol{\xi} = \boldsymbol{\nu} \times (\mathbf{n} \times \boldsymbol{\nu}), \quad \boldsymbol{\gamma} = (\boldsymbol{\alpha} \times \boldsymbol{\beta}) \times \dot{\mathbf{n}}, \quad \boldsymbol{\delta} = \beta_0 \boldsymbol{\alpha} - \alpha_0 \boldsymbol{\beta}$$

We will evaluate the derivative of the left-hand side of (5.2) by virtue of Eq. (5.1)

$$(\boldsymbol{\nu} \cdot \boldsymbol{\gamma})(\boldsymbol{\nu} \cdot \boldsymbol{\nu}) + (\mathbf{m} \cdot \boldsymbol{\nu})[\boldsymbol{\nu} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})] - (\boldsymbol{\nu} \cdot \boldsymbol{\nu})[\mathbf{m} \cdot (\boldsymbol{\alpha} \times \boldsymbol{\beta})] + \boldsymbol{\delta} \cdot (\boldsymbol{\xi} + \mathbf{m} \times \boldsymbol{\nu}) = 0 \tag{5.3}$$

We will require that relation (5.3) should be an identity in terms of the variables ν_1, ν_2, ν_2, and we will consider terms in ν_i of the highest powers, which give the first term. Since this term is equal to zero for all values of ν_i, then $\boldsymbol{\gamma} = 0$ or $\boldsymbol{\alpha} \times \boldsymbol{\beta} = \mu_0 \mathbf{n}$. On the basis of this condition, we convert Eq. (5.3) to the form

$$\mu_0 (\mathbf{m} \times \boldsymbol{\nu}) \cdot (\boldsymbol{\nu} \times \mathbf{n}) + \boldsymbol{\delta} \cdot (\boldsymbol{\xi} + \mathbf{m} \times \boldsymbol{\nu}) = 0 \tag{5.4}$$

If we put $\alpha_0 = \beta_0 = 0$ in Eq. (5.4), the vector $\boldsymbol{\delta} = 0$, and hence the condition $(\mathbf{m} \times \boldsymbol{\nu}) \cdot (\boldsymbol{\nu} \times \mathbf{n}) = 0$ must be satisfied. We will put $\boldsymbol{\nu} = x\mathbf{m} + y\mathbf{n}$ in this relation, where x and y are variables. We then obtain that $\mathbf{m} = x_* \mathbf{n}$ (x_* is a parameter). But, when this condition applies, we have the contradictory equality $(\mathbf{m} \times \boldsymbol{\nu}) \cdot (\mathbf{n} \times \boldsymbol{\nu}) = x_* (\mathbf{n} \times \boldsymbol{\nu})^2 = 0$. Consequently, we must put $\alpha_0^2 + \beta_0^2 \neq 0$ in integral (5.2).

In relation (5.4) consider terms linear in ν_i, which contain the term $\boldsymbol{\delta} \cdot (\mathbf{m} \times \boldsymbol{\nu})$. Since it must be identically equal to zero with respect to ν_i ($i = 1, 2, 3$), we obtain the condition $\boldsymbol{\delta} = \mu_0^* \mathbf{m}$. As a consequence of this equality, relation (5.4) takes the form

$$(\mu_0 - \mu_0^*)(\mathbf{m} \times \boldsymbol{\nu}) \cdot (\mathbf{n} \times \boldsymbol{\nu}) = 0$$

Hence it follows that $\mu_0^* = \mu_0$, and therefore the necessary conditions for integral (5.2) to exist in system (5.1) are

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0, \quad \boldsymbol{\alpha} \times \boldsymbol{\beta} = \mu_0 \mathbf{n}, \quad \beta_0 \boldsymbol{\alpha} - \alpha_0 \boldsymbol{\beta} = \mu_0 \mathbf{m} \tag{5.5}$$

System (5.5) has a non-trivial solution in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ only when $\mathbf{m} \cdot \mathbf{n} = 0$. Without loss of generality, in (5.5) we will put

$$\beta_0 = 0, \ \alpha_0 = m^2, \ \boldsymbol{\alpha} = \mathbf{n} \times \mathbf{m}, \ \boldsymbol{\beta} = -\frac{\mu_0}{m^2} \mathbf{m}$$

i.e. integral (5.2) is such that

$$\frac{m^2 + (\mathbf{n} \times \mathbf{m}) \cdot \boldsymbol{\nu}}{(\mathbf{m} \cdot \boldsymbol{\nu})} = l_0; \ \mathbf{n} \cdot \mathbf{m} = 0 \tag{5.6}$$

The condition for integral (5.6) to exist, expressed on the basis of the notation (3.2)–(3.4) in terms of the parameter of problem (1.1), leads to the following constraint

$$\lambda_1 a_2 c_3 - \lambda_2 a_2 b_3 + \lambda_3 a_3 b_2 = 0 \tag{5.7}$$

where b_2, b_3 and c_3 are given by relations (4.1).

We will investigate the solution of Eq. (4.4) when $\mathbf{m} \cdot \mathbf{n} = 0$. In Eq. (4.4) we will put

$$\mathbf{e} = \mu_1 \mathbf{n} + \mu_2 \mathbf{m} + \mu_3 (\mathbf{n} \times \mathbf{m})$$

and we will consider the equality $|\mathbf{e}| = 1$. Then, finding the coefficients μ_i, we have

$$\mathbf{e} = \frac{1}{m^2}(\mathbf{m} \times \mathbf{n} - a_1 b_1 \mathbf{m}); \ m^2 = a_1^2 b_1^2 + n^2 \tag{5.8}$$

Consequently, if the parameters of problem (1.1) and the parameters of relations (1.3), in addition to condition (5.7), satisfy the equality

$$\sum a_i^2 \lambda_i^2 = a_1^2 \sum b_i^2 + a_2^2 c_3^2 \tag{5.9}$$

Eq. (4.4) has the solution (5.8), and the motion of the gyrostat possesses the property of isoconicity. In view of the linearity of relation (5.7) with respect to λ_i, the system of equations (5.7), (5.9) is solvable for the parameters λ_i.

6. INTEGRATION OF EQ. (5.1)

For convenience we will change to new variables and parameters in Eq. (5.1). We put

$$\mathbf{n}_0 = \mathbf{n}/|\mathbf{n}|, \quad \mathbf{m}_0 = \mathbf{m}/|\mathbf{n}|, \quad \tau = |\mathbf{n}|t$$

We will denote differentiation with respect to τ by a prime. It then follows from Eq. (5.1) that

$$\boldsymbol{\nu}' = \mathbf{m}_0 \times \boldsymbol{\nu} + \mathbf{n}_0 - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}_0) \tag{6.1}$$

where we have taken into account the integral relation $\boldsymbol{\nu} \cdot \boldsymbol{\nu} = 1$, i.e. the integration is carried out over the Poisson sphere.

We will transform Eq. (6.1) in the general case, i.e. ignoring the condition $\mathbf{n}_0 \cdot \mathbf{m}_0 = 0$. We introduce the vector $\mathbf{d} = \mathbf{m}_0 - (\mathbf{n}_0 \cdot \mathbf{m}_0)\mathbf{n}_0$, orthogonal to $\mathbf{n}_0$. Equation (6.1) takes the form

$$\boldsymbol{\nu}' = \mathbf{d} \times \boldsymbol{\nu} + (\mathbf{n}_0 \cdot \mathbf{m}_0)(\mathbf{n}_0 \times \boldsymbol{\nu}) + \mathbf{n}_0 - \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{n}_0) \tag{6.2}$$

We will denote the components of the vector $\boldsymbol{\nu}$ in the basis $\mathbf{n}_0$, $\mathbf{d}$, $\mathbf{n}_0 \times \mathbf{d}$ by u, v and w. It then follows from Eq. (6.2) that

$$u' = d^2 w + 1 - u^2, \quad v' = -uv - (\mathbf{n}_0 \cdot \mathbf{m}_0)w, \quad w' = -u(1+w) + (n_0 \cdot \mathbf{m}_0)v \tag{6.3}$$

On the basis of the condition $|\boldsymbol{\nu}| = 1$ we conclude that the variables u, v and w satisfy the invariant relation

$$u^2 + d^2(v^2 + w^2) = 1 \tag{6.4}$$

In the general case, system (6.3) cannot be integrated. But in the case when a fractionally linear integral exists in it (see Section 5), this can be done, since when $\mathbf{n} \cdot \mathbf{m} = 0$ or $\mathbf{n}_0 \cdot \mathbf{m}_0 = 0$, Eqs (6.3) can be simplified and become

$$u' = m_0^2 w + 1 - u^2, \quad v' = -uv, \quad w' = -u(1+w) \tag{6.5}$$

and allow of the first integral

$$(w+1)/v = g_0 \tag{6.6}$$

where g_0 is an arbitrary constant. Since the variable u, v and w are connected by relation (6.4), in which we must put $d = m_0$, the integration of system (6.5), when equalities (6.4) and (6.6) hold, reduces to integration of the equation

$$\theta' = -\left(\frac{m_0 g_0}{h_0^2} + R_0 \sin\theta\right); \quad h_0^2 = 1 + g_0^2 \tag{6.7}$$

In obtaining this equation we used the replacement

$$u = R_0 \cos\theta, \quad v = \frac{g_0}{h_0^2} + \frac{R_0}{m_0 h_0}\sin\theta, \quad w = -\frac{1}{h_0^2} + \frac{g_0 R_0}{m_0 h_0}\sin\theta \left(R_0^2 = 1 - \frac{m_0^2}{h_0^2}\right) \tag{6.8}$$

where the constant g_0 must satisfy the condition $g_0^2 > m_0^2 - 1$.

From Eq. (6.7) we have

$$\theta = 2\,\mathrm{arctg}\left(\frac{1}{p_0}\sqrt{p_0^2 - q_0^2}\,\mathrm{tg}\frac{\sqrt{p_0^2 - q_0^2}}{2}(\tau - \tau_0) - q_0\right) \tag{6.9}$$

$$p_0 = -\frac{m_0 g_0}{h_0}, \quad q_0 = R_0$$

(we confine ourselves to the case when $p_0^2 - q_0^2 > 0$). We find the components of the vector $\boldsymbol{\nu}$ using the basis $\mathbf{n}_0$, $\mathbf{m}_0$, $\mathbf{n}_0 \times \mathbf{m}_0$

$$\nu_1 = n_0^{(1)} u + m_0^{(1)} v + (n_0^{(2)} m_0^{(3)} - n_0^{(3)} m_0^{(2)}) w \quad (1\ 2\ 3) \tag{6.10}$$

where $n_0^{(i)}$, $m_0^{(i)}$ ($i = 1, 2, 3$) are the components of the vectors $\mathbf{n}_0$ and $\mathbf{m}_0$ respectively.

By substituting expressions (6.9) into (6.8) we can determine the functions $u = u(\tau)$, $v = v(\tau)$, $w = w(\tau)$, by means of which we find the fundamental variables of the problem from Eqs (6.10) and (1.3).

The form of the angular velocity vector

$$\boldsymbol{\omega} = -\mathbf{m} + \mathbf{n} \times \boldsymbol{\nu} \quad (\mathbf{m} \cdot \mathbf{n} = 0) \tag{6.11}$$

is of interest.

7. THE CASE $m \times \mathbf{n} = 0$

In Sections 5 and 6 we considered the fractionally linear first integrals of system (5.1) and we showed that they exist provided the condition $\mathbf{m} \cdot \mathbf{n} = 0$ is satisfied. Since, in general, system (5.1) is not integrable, its integration, not based on the existence of algebraic integrals, is of interest [6].

Suppose that, in Eq. (6.1), the parameters of the problem are such that

$$\mathbf{m}_0 = a_0 \mathbf{n}_0 \tag{7.1}$$

(i.e. $\mathbf{m} \times \mathbf{n} = 0$ for Eqs (5.1)). In this case the conditions imposed on the parameters λ_i are such that

$$a_1\lambda_1 = -a_0 a_2 c_3, \quad a_2\lambda_2 = a_0 a_1 b_3, \quad a_3\lambda_3 = -a_0 a_1 b_2 \tag{7.2}$$

In relations (7.1) and (7.2) a_0 is a constant. The special case when $b_3 = c_3 = 0$ was considered previously [6].

Instead of the variables ν_i we will introduce the new variables p_i

$$\nu_l = n_0^{(l)} p_1 + (-1)^l n_0^{(3-l)} n_*^{-1} p_2 - n_0^{(l)} n_0^{(3)} n_*^{-1} p_3, \quad l = 1, 2$$
$$\nu_3 = n_0^{(3)} p_1 + n_* p_3 \tag{7.3}$$

As a result of replacement (7.3) the system of equations which follow from system (6.1) when condition (7.1) is satisfied can be reduced to the form

$$p_1' = 1 - p_1^2, \quad p_2' = -p_1 p_2 - a_0 p_3, \quad p_3' = a_0 p_2 - p_1 p_3 \tag{7.4}$$

System (7.4) can be integrated in terms of elementary functions

$$p_1 = \operatorname{th}(\tau + \tau_0), \quad p_2 = \frac{\cos(a_0\tau + \varphi_0)}{\operatorname{ch}(\tau + \tau_0)}, \quad p_3 = \frac{\sin(a_0\tau + \varphi_0)}{\operatorname{ch}(\tau + \tau_0)} \tag{7.5}$$

where φ_0 is an arbitrary constant. By successive substitution of expressions (7.5) into Eqs (7.3) and of the expressions obtained into relations (1.3), we obtain expressions for all the variables of the problem.

It follows from relations (7.3) and (7.5) that, when the variable τ is eliminated in expressions (7.6), we arrive at the first integral of Eq. (6.1), which has a transcendental form. The angular velocity vector of the gyroscope is found by analogy with expressions (4.2) and (6.11)

$$\boldsymbol{\omega} = a_0(-\mathbf{n} + \mathbf{n} \times \mathbf{v})$$

We will investigate the conditions for Eq. (4.4) to be solvable. By virtue of condition (7.1) the initial vectors are also collinear: $\mathbf{m} = a_0\mathbf{n}$. The solution of Eq. (4.4) is

$$\mathbf{e} = \mathbf{n}/|\mathbf{n}|, \quad a_0 = -a_1 b_1/|\mathbf{n}|$$

Consequently, if the parameter a_0 in solution (7.5) takes the value indicated above, the motion of the gyrostat will be isoconic.

Hence, we have shown that system (3.1) is integrable in four cases

$$1)\ G^- = 0,\ 2)\ G^+ = 0,\ \mathbf{m} = \mathbf{0},\ 3)\ G^+ = 0,\ \mathbf{m}\cdot\mathbf{n} = 0,\ 4)\ G^+ = 0,\ \mathbf{m}\times\mathbf{n} = 0$$

In the first case the additional integral of Poisson's equations has a polynomial form, in the second and third it has a fractionally linear form, and in the fourth it has a transcendental form.

As already noted above, other versions of the integrability of Poisson's equations in the case when three linear invariant relations exist in system (1.1) have been mentioned previously [1–3]. Chaplygin's version is of interest in the fact that G^+ and G^- do not vanish simultaneously. In the notation of this paper, this case is as follows: $g_{ij}^+ = 0$ $(i \neq j)$, $\mathbf{m} = 0$, $\mathbf{n} = (0, 0, n_3)$, but $g_{22}^+ \neq 0$, $g_{33}^+ \neq 0$.

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